

GOWERS NORMS FOR THE THUE-MORSE AND RUDIN-SHAPIRO SEQUENCES

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ABSTRACT. We estimate Gowers uniformity norms for some classical automatic sequences, such as the Thue-Morse and Rudin-Shapiro sequences. The methods can also be extended to other automatic sequences. As an application, we asymptotically count arithmetic progressions in the set of integers $\leq N$ where the Thue-Morse (resp. Rudin-Shapiro) sequence takes the value $+1$.

1. INTRODUCTION

The Thue-Morse sequence is among the simplest automatic sequences. It can be described by the recursive relations:

$$t(0) = 1, \quad t(2n) = t(n), \quad t(2n+1) = -t(n),$$

or by explicit formula $t(n) = (-1)^{s_2(n)}$, where $s_2(n)$ denotes the sum of digits of n base 2. Arguably, the Thue-Morse sequence is very structured — in particular, its subword complexity (i.e. number of distinct subsequences of a given length) has linear rate of growth (this is a general feature of automatic sequences). On the other hand, there are also ways in which it can be construed as pseudorandom.

Mauduit and Sarközy [MS98] studied several measures of pseudorandomness for the Thue-Morse sequence, and showed that t is highly uniform according to some but not all of those measures. In particular, it is shown that for any positive integers a, b, M, N with $a(M-1) + b < N$, we have

$$\sum_{n=0}^{M-1} t(an+b) = O(N^{\log 3 / \log 4}),$$

where the implied constant is absolute. However, for all sufficiently large integers N , there exist positive integers M and h with $M+h \leq N$ such that

$$\left| \sum_{n=0}^{M-1} t(n)t(n+h) \right| \geq cN,$$

where $c > 0$ is an absolute constant. (We may take $c = 1/24$.) Thus, the Thue-Morse sequence does not correlate with arithmetic progressions but can have some large self-correlations.

Here, we consider a different notion of pseudorandomness related to Gowers uniformity norms.

Definition 1.1. Fix $s \in \mathbb{N}$. For $N \in \mathbb{N}$ and $f: [N] \rightarrow \mathbb{R}$, the s -th Gowers uniformity norm of f is defined as

$$(1) \quad \|f\|_{U^s[N]}^{2^s} := \mathbb{E}_{n, \mathbf{h}} \prod_{\omega \in \{0,1\}^s} f(n + \omega \cdot \mathbf{h})$$

where $\omega \cdot \mathbf{h} = \sum_{i=1}^s \omega_i h_i$, and the expectation is taken over all $n \in \mathbb{Z}$, $\mathbf{h} \in \mathbb{Z}^s$ for which the cube $\{n + \omega \cdot \mathbf{h} \mid \omega \in \{0,1\}^s\}$ is contained in $[N]$.

(We consistently use boldface letter \mathbf{x} to denote vector with coordinates (x_i) ; and also write $|\mathbf{x}| := \sum_i x_i$. By $[N]$ we denote the interval $\{0, 1, \dots, N\}$.)

A sequence is (informally) said to be Gowers uniform of order s if its $U^s[N]$ -norm is small. We show that t indeed is highly Gowers uniform.

Theorem A. *Let $t: \mathbb{N}_0 \rightarrow \{\pm 1\}$ denote the Thue-Morse sequence. For any $s \in \mathbb{N}$, there exists $c = c(s) > 0$ such that $\|t\|_{U^s[N]} = O(N^{-c})$ as $N \rightarrow \infty$.*

A slightly more complicated sequence we deal with carries the name Rudin-Shapiro. It is recursively given by

$$r(0) = 0, \quad r(2n) = r(n), \quad r(4n+1) = r(n), \quad r(4n+3) = -r(2n+1),$$

or explicitly by $r(n) = (-1)^{f_{11}(n)}$, where $f_{11}(n)$ denotes the number of times the pattern 11 appears in the binary expansion of n .

Much like in the case of the Thue-Morse sequence, various pseudorandomness properties of the Rudin-Shapiro sequence are studied in [MS98]. Specifically, it is again the case that r does not correlate with arithmetic progressions, but has large self-correlations.

Theorem B. *Let $r: \mathbb{N}_0 \rightarrow \{\pm 1\}$ denote the Rudin-Shapiro sequence. For any $s \in \mathbb{N}$, there exists $c = c(s) > 0$ such that $\|r\|_{U^s[N]} = O(N^{-c})$ as $N \rightarrow \infty$.*

While we focus our attention on these two specific sequences, many of the observations apply to more general automatic sequences. A sequence $a: \mathbb{N}_0 \rightarrow \mathbb{C}$ is k -automatic if $a(n)$ can be computed by a finite automaton taking k -ary expansion of n on input. For comprehensive background, we refer to [AS03].

We close by remarking that a key reason for interest in the Gowers uniformity norms is their usefulness in counting linear patterns. In particular, as an immediate corollary of Theorem A (resp. B) we conclude that the number of k -term arithmetic progressions in $\{n \in [N] \mid t(n) = +1\}$ (resp. $\{n \in [N] \mid r(n) = +1\}$) is $N^2/2k + O(N^{2-c})$. For relevant theory, we refer to [Gre15] and [Tao12].

Acknowledgements. The author is grateful to Tanja Eisner for posing this question and for her hospitality during his stay in Leipzig when the work on this project began; to Jakub Byszewski for many long and productive discussions; and to Ben Green for his encouragement and useful advice.

2. THUE-MORSE SEQUENCE

The purpose of this section is to prove Theorem A, asserting that the uniformity norms of the Thue-Morse sequence $t(n) = (-1)^{s_2(n)}$ are small. Throughout this section, let $s \in \mathbb{N}_{\geq 2}$ be fixed. It will be convenient to study somewhat more general averages

$$(2) \quad A(L, \mathbf{r}) := \mathbb{E}_{n, \mathbf{h}} \prod_{\omega \in \{0,1\}^s} t(n + \omega \cdot \mathbf{h} + r_\omega),$$

where n, \mathbf{h} parametrize the cubes $\{n + \omega \cdot \mathbf{h} \mid \omega \in \{0, 1\}^s\} \subset [2^L]$, $L \in \mathbb{N}_0$ and $\mathbf{r} = (r_\omega)_{\omega \in \{0, 1\}^s}$ with $r_\omega \in \mathbb{Z}$. Note that if $\mathbf{r} = \mathbf{0}$, then (2) defines $\|t\|_{U^s[2^L]}^{2^s}$.

Lemma 2.1. *The averages $A(L, \mathbf{r})$ satisfy the recursive relation*

$$(3) \quad A(L, \mathbf{r}) = (-1)^{|\mathbf{r}|} \mathbb{E}_{\mathbf{e}} A(L-1, \mathbf{r}'(\mathbf{e})) + O(2^{-L}),$$

where the average is taken over $\mathbf{e} = (e_i)_{i=0}^s \in \{0, 1\}^{s+1}$ and $\mathbf{r}'(\mathbf{e})$ is given by

$$(4) \quad r'_\omega(\mathbf{e}) = \left\lfloor \frac{r_\omega + (1, \omega) \cdot \mathbf{e}}{2} \right\rfloor = \left\lfloor \frac{r_\omega + e_0 + \sum_{i=1}^s \omega_i e_i}{2} \right\rfloor.$$

Proof. The cubes $\{n + \omega \cdot \mathbf{h} \mid \omega \in \{0, 1\}^s\}$ contained in $[2^L]$ are (almost) precisely the cubes of the form $\{2n' + 2\omega \cdot \mathbf{h}' + (1, \omega) \cdot \mathbf{e} \mid \omega \in \{0, 1\}^s\}$ where $\mathbf{e} \in \{0, 1\}^{s+1}$ and $\{n' + \omega \cdot \mathbf{h}' \mid \omega \in \{0, 1\}^s\}$ is a cube contained in $[2^{L-1}]$, except for a proportion of $O(2^{-L})$ of exceptions. It follows that

$$\begin{aligned} A(L, \mathbf{r}) &= \mathbb{E}_{\mathbf{e}} \mathbb{E}_{n', \mathbf{h}'} \prod_{\omega \in \{0, 1\}^s} t(2n' + 2\omega \cdot \mathbf{h}' + (1, \omega) \cdot \mathbf{e} + r_\omega) + O(2^{-L}) \\ &= \mathbb{E}_{\mathbf{e}} \mathbb{E}_{n', \mathbf{h}'} \prod_{\omega \in \{0, 1\}^s} (-1)^{(1, \omega) \cdot \mathbf{e} + r_\omega} t(n' + \omega \cdot \mathbf{h}' + r'_\omega(\mathbf{e})) + O(2^{-L}) \\ &= \mathbb{E}_{\mathbf{e}} (-1)^{S(\mathbf{e})} A(L-1, \mathbf{r}'(\mathbf{e})) + O(2^{-L}), \end{aligned}$$

where $\mathbf{r}'(\mathbf{e})$ is defined by (4) and $S(\mathbf{e}) = \sum_\omega r_\omega + 2^s e_0 + 2^{s-1} \sum_{i=1}^s e_i \equiv |\mathbf{r}| \pmod{2}$ (expectation is over $\mathbf{e} \in \{0, 1\}^{s+1}$ and $\{n' + \omega \cdot \mathbf{h}' \mid \omega \in \{0, 1\}^s\} \subset [2^{L-1}]$). \square

Lemma 2.1 motivates us to introduce a random walk \mathcal{W}_{TM} on a directed graph $G = (V, E)$ defined as follows. The set of vertices is $V = V_+ \cup V_-$ where $V_\pm = \{(\mathbf{r}, \pm 1) \mid \mathbf{r} \in \mathbb{Z}^{\{0, 1\}^s}\}$. The transition probabilities are given by

$$(5) \quad P\left((\mathbf{r}, \pm 1); (\mathbf{r}', \pm(-1)^{|\mathbf{r}|})\right) = \mathbb{P}_{\mathbf{e}}(\mathbf{r}'(\mathbf{e}) = \mathbf{r}'),$$

where $\mathbf{e} = (e_i)_{i=0}^s$ is uniformly distributed in $\{0, 1\}^{s+1}$ and $\mathbf{r}'(\mathbf{e})$ is given by (4). (By convention, the two occurrences of the symbol \pm both denote the same sign.) The remaining transition probabilities (i.e. those where the signs do not agree) are declared to be identically 0.

The set E of (directed) edges of G consists of the pairs $(v, v') \in V^2$ with $P(v, v') > 0$; hence the edge $(\mathbf{r}, \pm 1) \rightarrow (\mathbf{r}', \pm(-1)^{|\mathbf{r}|})$ is present if and only if there exists $\mathbf{e} \in \{0, 1\}^{s+1}$ such that $\mathbf{r}'(\mathbf{e}) = \mathbf{r}'$ (with $\mathbf{r}'(\mathbf{e})$ given by (4)). We will be particularly interested in the graph G_0 consisting of the vertices V_0 reachable from the initial vertex $(\mathbf{0}, +1)$.

We note that G comes with a natural symmetry $\mathcal{R}: V \rightarrow V$ given by $(\mathbf{r}, \pm 1) \mapsto (\mathbf{r}, \mp 1)$. We have $\mathcal{R}(\mathcal{R}(v)) = v$ and $P(\mathcal{R}(v), \mathcal{R}(v')) = P(v, v')$ for all $v, v' \in V$. In particular, \mathcal{R} preserves the edges of G .

Denote further by $P^{(l)}(v, v')$ the probability of reaching vertex v' after l steps, starting from v . Iterating Lemma 2.1 we obtain the following formula.

Corollary 2.2. The averages $A(L, \mathbf{r})$ satisfy for any $l < L$ the recursive relation (6)

$$A(L, \mathbf{r}) = \sum_{\mathbf{r}'} A(L-l, \mathbf{r}'(\mathbf{e})) \times \left(P^{(l)}((\mathbf{r}, +1), (\mathbf{r}', +1)) - P^{(l)}((\mathbf{r}, +1), (\mathbf{r}', -1)) \right) + O(2^{-(L-l)}),$$

where the sum runs over all \mathbf{r}' such that one of $(\mathbf{r}', +1)$ or $(\mathbf{r}', -1)$ is in reachable from $(\mathbf{r}, +1)$.

Recall that a directed graph is strongly connected if there exists a directed path from any vertex to any other vertex. A graph is aperiodic if the greatest common divisor of all cycles present in the graph equals 1.

Proposition 2.3. Let G_0 be the graph constructed above. Then G_0 is finite, strongly connected, aperiodic, and preserved by \mathcal{R} .

Proof. Aperiodicity follows immediately from the observation that G_0 contains a loop at $(\mathbf{0}, +1)$, corresponding to taking $\mathbf{e} = \mathbf{0} \in \{0, 1\}^{s+1}$ in (4).

If G contains an edge from $v = (\mathbf{r}, \pm 1)$ to $v' = (\mathbf{r}', \pm 1)$ then $r'_\omega = r'_\omega(\mathbf{e}) = \left\lfloor \frac{r_\omega + (1, \omega) \cdot \mathbf{e}}{2} \right\rfloor$ for some $\mathbf{e} \in \{0, 1\}^{s+1}$ and in particular $0 \leq (1, \omega) \cdot \mathbf{e} \leq |\omega| + 1$. An elementary inductive argument now shows that if $(\mathbf{r}, \pm 1) \in V_0$ then $0 \leq r_\omega \leq |\omega|$, which proves finiteness.

Similarly, taking $\mathbf{e} = \mathbf{0} \in \{0, 1\}^{s+1}$, we see that any vertex $v = (\mathbf{r}, \pm 1)$ has an edge to some $v' = (\mathbf{r}', \pm 1)$ with $|\mathbf{r}'| \leq |\mathbf{r}|/2$. Repeating this argument, we may find a path from any $v \in V_0$ to one of $(\mathbf{0}, +1)$, $(\mathbf{0}, -1)$. Thus, to prove that G_0 is strongly connected, it will suffice to show that there exists a path from $(\mathbf{0}, -1)$ to $(\mathbf{0}, +1)$, which (in light of symmetry) is equivalent to $(\mathbf{0}, -1) \in V_0$. Since G is symmetric under \mathcal{R} , this will also imply that $\mathcal{R}(V_0) = V_0$.

It remains to show that $(\mathbf{0}, -1) \in V_0$. We do this by explicitly constructing the path from $(\mathbf{0}, +1)$ to $(\mathbf{0}, -1)$. Let $\mathbf{r}^{(0)} = \mathbf{0}$, and let $\mathbf{r}^{(j)} = (r_\omega^{(j)})$ for $j = 1, 2, \dots, s$ be given by

$$r_\omega^{(j)} = \begin{cases} 1 & \text{if } \omega_1 = \omega_2 = \dots = \omega_i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each $j = 0, 1, \dots, s-1$, there is an edge from $(\mathbf{r}^{(j)}, +1)$ to $(\mathbf{r}^{(j+1)}, +1)$, which corresponds to taking $e_0 = \delta_{0,j}$ and $e_i = \delta_{i,j+1}$ for $i = 1, 2, \dots, s$. (Here, $\delta_{i,j}$ denotes the Kronecker symbol equal to 1 if $i = j$ and 0 otherwise.) Finally, there is an edge from $(\mathbf{r}^{(s)}, +1)$ to $(\mathbf{0}, -1)$, which corresponds to taking $\mathbf{e} = \mathbf{0}$. \square

Corollary 2.4. There exists a constant $c > 0$ such that $A(L, \mathbf{0}) = O(e^{-cL})$.

Proof. Because G is aperiodic and strongly connected, the Frobenius-Perron Theorem implies that there exists a stationary distribution $\pi: V_0 \rightarrow [0, 1]$ such that for each $v, v' \in V_0$ we have $P^{(l)}(v, v') \rightarrow \pi(v')$ with exponential convergence rate:

$$(7) \quad \max_{v, v' \in V_0} \left| P^{(l)}(v, v') - \pi(v') \right| = O(e^{-cl})$$

for some $c > 0$. Because $\mathcal{R}(V_0) = V_0$, by symmetry we have $\pi(\mathcal{R}(v)) = \pi(v)$ for all $v \in V_0$. Combining this with (7), we obtain

$$(8) \quad \max_{v, v'} \left| P^{(l)}(v, v') - P^{(l)}(v, \mathcal{R}(v')) \right| = O(e^{-cl}).$$

Using this estimate and the trivial bound $|A(L, \mathbf{r})| \leq 1$ in (6), we arrive at

$$(9) \quad A(L, \mathbf{0}) = O(e^{-cl} + 2^{-(L-l)}).$$

It remains to put $l = L/2$ (say) to conclude that $A(L, \mathbf{0}) = O(e^{-c'L})$. \square

Remark 2.5. Computation of the constant c in the argument above is essentially equivalent to computing the spectral gap for the matrix $(P(v, v'))_{v, v' \in V_0}$. In particular, for fixed s , this is a computationally tractable problem.

Proof of Theorem A. Split $[N]$ into intervals $I(j) = [m_j 2^{L_j}, (m_j + 1) 2^{L_j}]$ where $L_1 > L_2 > \dots$. We then have

$$(10) \quad \|t\|_{U^s[N]} \ll \sum_j \|1_{I(j)} t\|_{U^s[N]}.$$

It is easy to see that the cubes $\{n + \omega \cdot \mathbf{h} \mid \omega \in \{0, 1\}^s\} \subset I(j)$ are precisely the translations of the cubes $\{n + \omega \cdot \mathbf{h} \mid \omega \in \{0, 1\}^s\} \subset [2^{L_j}]$ translated by $2^{L_j} m_j$. Hence, $\|1_{I(j)} t\|_{U^s[N]} \ll \left(\frac{2^{L_j}}{N}\right) \|t\|_{U^s[2^{L_j}]}$. Inserting this into (10) and applying Corollary 2.4 yields the sought bound. \square

3. RUDIN-SHAPIRO SEQUENCE

We now move on to the Rudin-Shapiro sequence $r(n) = (-1)^{f_{11}(n)}$, and embark upon the proof of Theorem B. Our argument is similar to the one in Section 2, although slightly more technical. Complications arise because of the fact that the 2-kernel

$$\mathcal{N}_2(r) = \{n \mapsto r(2^l n + m) \mid 0 \leq m < 2^l\} = \{\pm r(n), \pm r(2n + 1)\}$$

contains other functions apart from $\pm r(n)$, which forces us to deal with averages more general than those in (2).

A key feature of $r(n)$ which allows our argument to work is that $\mathcal{N}_2(r)$ is symmetric, i.e. $\mathcal{N}_2(r) = -\mathcal{N}_2(r)$. Denote $\mathcal{N}_2^+(r) = \{r(n), r(2n + 1)\}$, and fix from now on the value of $s \in \mathbb{N}_{\geq 2}$. We will study the averages

$$(11) \quad A(L, \mathbf{a}, \mathbf{r}) := \mathbb{E}_{n, \mathbf{h}} \prod_{\omega \in \{0, 1\}^s} a_\omega(n + \omega \cdot \mathbf{h} + r_\omega),$$

where $\{n + \omega \cdot \mathbf{h} \mid \omega \in \{0, 1\}^s\} \subset [2^L]$, $L \in \mathbb{N}_0$, $\mathbf{a} = (a_\omega)_{\omega \in \{0, 1\}^s}$ with $a_\omega \in \mathcal{N}_2^+(a)$ and $\mathbf{r} = (r_\omega)_{\omega \in \{0, 1\}^s}$ with $r_\omega \in \mathbb{Z}$.

We record a recurrence relation analogous to 2.1. Now, it will be more convenient to consider l consecutive steps.

Lemma 3.1. *For any l , the averages $A(L, \mathbf{a}, \mathbf{r})$ obey the recursive relation:*

$$(12) \quad A(L, \mathbf{a}, \mathbf{r}) = \mathbb{E}_{0 \leq \mathbf{e} < 2^l} A(L - l, \mathbf{a}'(\mathbf{e}), \mathbf{r}'(\mathbf{e})) + O(2^{-(L-l)}),$$

where the average is taken over $\mathbf{e} = (e_i)_{i=0}^{s-1} \in [2]^s$, $\mathbf{a}'(\mathbf{e})$ is given by

$$(13) \quad a'_\omega(\mathbf{e})(n) = a_\omega(2^l n + (r_\omega + (1, \omega) \cdot \mathbf{e} \bmod 2^l)),$$

and $\mathbf{r}'(\mathbf{e})$ is given by

$$(14) \quad r'_\omega(\mathbf{e}) = \left\lfloor \frac{r_\omega + (1, \omega) \cdot \mathbf{e}}{2^l} \right\rfloor.$$

Note that $\mathbf{a}'(\mathbf{e})$ and $\mathbf{r}'(\mathbf{e})$ depend also on l ; we suppress this dependence for the sake of readability.

The random walk \mathcal{W}_{RS} on $G = (V, E)$ we associate to the averages $A(L, \mathbf{a}, \mathbf{r})$ is now constructed as follows. The set of vertices V consists of triples $(\mathbf{a}, \mathbf{r}, \pm 1)$, where $a_\omega \in \mathcal{N}_2^+(r)$ and $r_\omega \in \mathbb{Z}$. Using the fact that $\mathcal{N}_2(r) = \mathcal{N}_2^+(r) \cup (-\mathcal{N}_2^+(r))$, we see that for any \mathbf{a} with $a_\omega \in \mathcal{N}_2(r)$, we can find $\bar{\mathbf{a}} = (\bar{a}_\omega)_{\omega \in \{0,1\}^s}$ with $\bar{a}_\omega \in \mathcal{N}_2^+(r)$ and $S = \pm 1$, such that $\prod_\omega a_\omega(x_\omega) = S \prod_\omega \bar{a}_\omega(x_\omega)$. For the sake of brevity, for $v = (\mathbf{a}, \mathbf{r}, S) \in V$, write $A(L, v) = SA(L, \mathbf{a}, \mathbf{r})$. Hence, we may find transition probabilities $P(v, v')$ for $v, v' \in V$ such that (12) for $l = 1$ is equivalent to

$$(15) \quad A(L, v) = \sum_{v' \in V} P(v, v') A(L-1, v') + O(2^{-L}).$$

The edge from v to v' is present in the edge set E of G if $P(v, v') > 0$.

More generally, we have

$$(16) \quad A(L, v) = \sum_{v' \in V} P^{(l)}(v, v') A(L-l, v') + O(2^{-(L-l)}),$$

where $P^{(l)}(v, v')$ denotes the probability of transition from v to v' in l steps. Accordingly, a path of length l from $v = (\mathbf{a}, \mathbf{r}, +1)$ to $v' = (\mathbf{a}', \mathbf{r}', S)$ exists if and only if the average $A(L-l, v)$ is present on the right hand side of (12), meaning that there exists $\mathbf{e} = (e_i)_{i=0}^k$, $0 \leq e_i < 2^l$, such that (with notation in Lemma 3.1) $A(L, \mathbf{a}'(\mathbf{e}), \mathbf{r}'(\mathbf{e})) \equiv A(L, v)$.

Following the same reasoning as before, we note that G has a natural symmetry $\mathcal{R}: V \rightarrow V$ given by $(\mathbf{a}, \mathbf{r}, S) \mapsto (\mathbf{a}, \mathbf{r}, -S)$, which preserves the transition probabilities. We will denote by V_0 the set of vertices reachable from the initial vertex $v_0 = ((r)_{\omega \in \{0,1\}^s}, \mathbf{0}, +1)$ and by G_0 the induced graph.

Proposition 3.2. Let G_0 be the graph constructed above. Then G_0 is finite, strongly connected, aperiodic, and preserved by \mathcal{R} .

Proof. Finiteness, aperiodicity, and strong connectedness follow from essentially the same argument as in Propositions 2.3. It remains to prove that $\mathcal{R}(v_0)$ is reachable from v_0 .

Pick any $l \geq s+2$, and $e_i = 2^{i-1}$ for $i = 1, 2, \dots, s$; we leave $0 \leq e_0 < 2^s$ undefined for the time being. It follows from Lemma 3.1 and subsequent discussion that G_0 contains a path of length l from v_0 to $v_1 = (\bar{\mathbf{a}}', \mathbf{r}', S)$, corresponding to the averages $A(L, \mathbf{a}'(\mathbf{e}), \mathbf{r}'(\mathbf{e}))$ (with the notation of Lemma 3.1 starting from $a_\omega = r$ and $\mathbf{r} = \mathbf{0}$). Here, $\bar{\mathbf{a}}' = (\bar{a}'_\omega) = (\pm a'_\omega(\mathbf{e}))$ with the sign chosen so that $\bar{a}'_\omega \in \mathcal{N}_2^+(r)$, meaning that

$$\bar{a}'_\omega(n) = \pm r(2^l n + (1, \omega) \cdot \mathbf{e}) = \pm r(n) r((1, \omega) \cdot \mathbf{e}) = r(n).$$

Above, we use that fact that $(1, \omega) \cdot \mathbf{e} < 2^{l-1}$. Next, $\mathbf{r}' = (r'_\omega) = \mathbf{r}'(\mathbf{e})$ is given by

$$r'_\omega = \left\lfloor \frac{(1, \omega) \cdot \mathbf{e}}{2^l} \right\rfloor = 0,$$

by virtue of the same estimate as before. Finally, S is chosen so that $A(L, v_1) = A(L, \mathbf{a}'(\mathbf{e}), \mathbf{r}'(\mathbf{e}))$, whence

$$S = \prod_{\omega \in \{0,1\}^s} r((1, \omega) \cdot \mathbf{e}) = \prod_{m=0}^{2^s-1} r(m + e_0).$$

Thus, v_1 is equal to $\mathcal{R}(v_0)$, provided that $S = S(e_0)$ defined above is equal to -1 , and it remains to find e_0 for which this is the case. In fact, it will suffice to show that $S(e_0)$ is not constant. Since $S(e_0 + 1)/S(e_0) = r(2^s + e_0)/r(e_0)$, we have $S(e_0 + 1) = -S(e_0)$ for any choice $2^{s-1} \leq e_0 < 2^s$, which finishes the argument. \square

Corollary 3.3. For any \mathbf{a}, \mathbf{r} with $a_\omega \in \mathcal{N}_2^+(r)$ and $r_\omega \in \mathbb{N}_0$, there exists a constant $c > 0$ such that $A(L, \mathbf{a}, \mathbf{r}) = O(e^{-cL})$.

Proof. When $a_\omega = r$ for all $\omega \in \{0, 1\}^s$ and $\mathbf{r} = \mathbf{0}$, then this follows by the same argument as in Corollary 2.4. The same argument also applies if $v_1 = (\mathbf{a}, \mathbf{r}, +1)$ is reachable from $v_0 = ((r)_{\omega \in \{0, 1\}^s}, \mathbf{0}, +1)$.

For the general case, note the set of vertices reachable from v_1 is finite. Moreover, from any vertex in V , there exists a path to v_0 or $\mathcal{R}(v_0)$ (it suffices to use edges corresponding to $\mathbf{e} = 0$ in (14)). Hence, there exists c_1 (dependent on v_1) such that the probability that the random walk starting at v_1 does not end in V_0 after $L/2$ steps is $O(e^{-c_1 L})$. Hence,

$$|A(L, v_1)| \leq \max_{v \in V_0} |A(L/2, v)| + O(e^{-c_1 L}),$$

and the sought bound follows. \square

Proof of Theorem B. Exact adaptation of the proof of Theorem A, but with application of Corollary 3.3 in place of 2.4. \square

4. CLOSING REMARKS

Our argument in Section 3 dealing with the Rudin-Shapiro sequence can be generalised to other automatic sequences. A crucial feature of the Rudin-Shapiro sequence which we exploited was the symmetry of the kernel, so let us restrict our attention to 2-automatic sequences $a(n)$ with $\mathcal{N}_2(a) = -\mathcal{N}_2(a)$.

The recursive relation analogous to (12) from Lemma 3.1 holds in full generality, and similarly the random walk can be constructed without any significant modifications. It remains true that the underlying graph is symmetric, and that the analogue of (15) holds. The endgame is also essentially the same, assuming that we can prove that the walk starting from the initial vertex has the properties mentioned in Proposition 3.2. (To avoid technical complications, assume that the set of vertices reachable from the origin is strongly connected.)

The key difficulty lies in the proof of the symmetry of the set of vertices reachable from the origin. This is essentially the same as saying that there exists two symmetric vertices, both of which are reachable. Proof of this can be carried out by a similar argument for several classes of sequences, such as $a(n) = (-1)^{f(n)}$ where $f(n)$ counts occurrences of a pattern π in the binary expansion of n (so $\pi = 1$ for Thue-Morse and $\pi = 11$ for Rudin-Shapiro; we require that π starts with 1).

Hence, as long as we are able to prove that the random walk associated to $a(n)$ and $s \in \mathbb{N}$ has the required properties, we may conclude that $\|a\|_{U^s[N]} = O(N^{-c})$ for a constant $c = c(a, s) > 0$. We remark that if $a(n)$ and s are given, this is a finite and computationally feasible check.

Conversely, one may ask about the minimal conditions under which it could be shown that $\|a\|_{U^s[N]} = o(1)$. It is well-known that a sequence with small uniformity norm cannot correlate with a polynomial phase, or indeed an s -step nilsequence. While it would be surprising to find an automatic sequence correlating with (say) a

quadratic phase $e^{2\pi i \alpha n^2}$ (with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$), it is certainly possible to have automatic sequences which correlate with periodic sequences. (In fact, in the situation above it is not hard to show that $\mathbb{E}_{n < N} a(n) e^{2\pi i \alpha n^2} = o(1)$ as $N \rightarrow \infty$.)

Motivated by our main theorems and the above discussion, we are led to suspect the following.

Conjecture. Let $a(n)$ be a 2-automatic sequence such that $\mathbb{E}_{n < N} a(qn + r) \rightarrow 0$ as $N \rightarrow \infty$ for any $q \in \mathbb{N}$, $r \in \mathbb{N}_0$. Then, $\|a\|_{U^s[N]} \rightarrow 0$ as $N \rightarrow \infty$ for any $s \in \mathbb{N}$.

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